Lab 4: Transmission Line

1 Introduction

In this experiment we will study the properties of a wave propagating in a periodic medium. Usually this takes the form of an array of masses and springs of the kind whose normal modes we have studied. However, the relative phases of the oscillations are important for this experiment, and phase is a bit difficult to measure with mechanical oscillators. Therefore we will use an electrical analog.

We will use a combination of inductors and capacitors, and will begin the discussion with a review of the time-dependent behavior of the voltages and currents. This will be extended from the simple case of a single LC circuit to a coupled array of them. It is called a "transmission line" because real electrical transmission lines have both capacitance and inductance. The capacitance can arise from the shield of a coaxial cable or from the ground of a power transmission line, and the inductance can arise from the geometry of the current-carrying wires.

2 Theory

Figure 1 shows a resonant LC circuit that can be analyzed simply using Kirchoff's loop law. This law is nothing special - it's simply a statement of conservation of electrical energy for the case of a closed path, and is simply written as $\sum_i V_i = 0$. It means that the sum of the voltages around any closed path is zero, and of course, this is required because transporting a charge q around a closed path shouldn't change its energy, and the energy change for each step of voltage ΔV_i is simply $\Delta E_i = q\Delta V_i$.

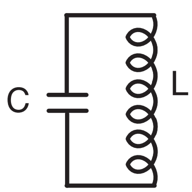

Figure 1: This is the simplest possible LC circuit. An oscillation can be started by putting some charge q on the capacitor. It will flow off the capacitor, and establish a magnetic field in the inductor. Once q decreases to zero, the current will be maintained by the collapsing field of the inductor until there is $-q$ on the capacitor, and then the cycle reverses. It's the same as a swinging pendulum or mass on a spring. In this case the current provides the inertia and the capacitor corresponds to gravity or the spring tension.

The voltage across a capacitor C carrying charge q is $V = q/C$, and the voltage across an inductor carrying a steady current i is zero. But if the current is time-dependent, then the voltage across an inductor L is $V = L(di/dt)$. Thus for the loop shown in Fig. 1 we use Kirchoff's law to find $L(d^2q/dt^2) + q/C = 0$ since $i \equiv (dq/dt)$. Except for the different names of the variables, this equation is identical to that of the harmonic oscillator we have studied in detail, and so we find $q = q_0\Re[e^{i(\omega_0 t+\phi)}]$ where $\omega_0 \equiv 1/\sqrt{LC}$ as one of the many possible solutions. Remember that $\Re[z]$ means the real part of z, and q is a real quantity.

Next we consider the arrangement of Fig. 2. We look at the loop around each unit, starting from the ground, up through one capacitor, through an inductor, and then down through the next capacitor. When the current i flowing in an inductor reaches the junction, some of it flows on to the capacitor and some flows through the next inductor. We can write Kirchoff's loop law, $\sum_i V_i = 0$, for the n^{th} LC loop as

$$\frac{q_{n-1}}{C} - L\frac{di_{n-1}}{dt} - \frac{q_n}{C} = 0 \quad \text{and for the next loop} \quad \frac{q_n}{C} - L\frac{di_n}{dt} - \frac{q_{n+1}}{C} = 0. \tag{1}$$

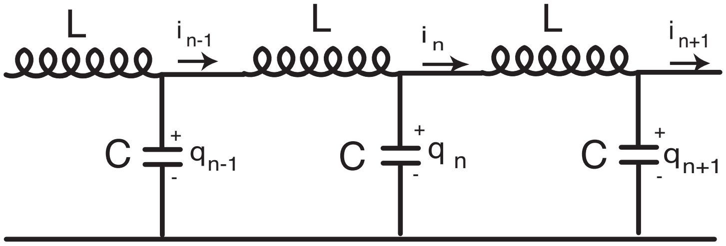

Figure 2: Schematic diagram of a three-unit LC transmission line. Because the capacitors and inductors are discrete instead of the continuous capacitance and inductance of a real transmision line, this is sometimes called a "lumped constant" transmission line. We'll avoid the term here because it's awkward, but you're quite likely to see it again.

We subtract the two equations of Eq. 1 from one another, multiply the result by C, and find $q_{n-1} - 2q_n + q_{n+1} = LC(d/dt)(i_{n-1} - i_n)$. Now the current divides at the junction so $i_{n-1} = i_n + C(dV_n/dt)$ because the current onto the n^{th} capacitor is simply $dq_n/dt = C(dV_n/dt)$. So we use this to replace $(i_{n-1} - i_n)$ with $C(dV_n/dt)$ and find

$$q_{n-1} - 2q_n + q_{n+1} = LC\frac{d^2q_n}{dt^2} = \frac{1}{\omega_0^2}\frac{d^2q_n}{dt^2}. \tag{2}$$

Equation 2 is yet another differential equation that we have no idea how to solve. So we try a solution similar to those we have tried before, namely $q_n = \Re[q_0 e^{i(\omega t + \phi_n)}]$. We really can't tell anything about q_{n-1} or q_{n+1} from this, but we might expect those to oscillate at the same frequency ω, but to have a different phase, somewhat like the multiple masses on the air track or the loops of a slinky.

Now we make a really daring insight. We suppose that the charge oscillating in each LC loop is the same as in any other loop, except for the different phase, and we also guess that the phase difference between any pair of adjacent loops is the same as the phase difference between any other adjacent pair of loops. That is, we'll assume that $\phi_n - \phi_{n-1}$ is the same as $\phi_{n+1} - \phi_n$, and call this simply ϕ. Then we go ahead and substitute our trial solution into Eq. 2, multiply both sides by $e^{-i\omega t}/q_0$, and find

$$e^{i\phi_{n-1}} + e^{i\phi_{n+1}} - 2e^{i\phi_n} = -\left[\frac{\omega}{\omega_0}\right]^2 e^{i\phi_n}, \text{ and } \phi = \phi_n - \phi_{n-1} = \phi_{n+1} - \phi_n \text{ yields } e^{i\phi} + e^{-i\phi} - 2 = -\left[\frac{\omega}{\omega_0}\right]^2. \tag{3}$$

The left side of the last relation in Eq. 3 is easily handled because $e^{i\phi} + e^{-i\phi} - 2 = [e^{i\phi/2} - e^{-i\phi/2}]^2 = [2i\sin(\phi/2)]^2$. So we get $[2i\sin(\phi/2)]^2 = -(\omega/\omega_0)^2$, and then taking the square root of both sides gives $2\sin(\phi/2) = \omega/\omega_0$ or $\omega = (2\omega_0)\sin(\phi/2)$.

Question 1: Show that $e^{i\phi} + e^{-i\phi} - 2 = -[2\sin(\phi/2)]^2$

Thus our result is consistent with our contention at the beginning of this calculation. We find an acceptable solution is that the oscillations in all the loops are the same except for the phase shifts. Furthermore, after the phase shifts have accumulated a total of 2π they repeat, and the characteristic number of LC units for this accumulation is defined to be N. Then we can say that each oscillator's phase shift is $\phi = 2\pi/N$ so that $\phi_n = 2\pi n/N$. If the line were a series of masses connected by springs with a their separation, or a crystal with lattice constant a, then $\lambda \equiv Na$ would be a characteristic repeat length.

Then the equation above for ω gives

$$\omega = (2\omega_0)\sin(\pi/N) = (2\omega_0)\sin(\phi/2). \qquad (4)$$

Wave propagation in periodic structures is most commonly studied in crystals where the lattice spacing is a and $\phi \equiv ka$ so that $\sin(\pi/N) = \sin(ka/2)$. The we find $k \equiv 2\pi/\lambda = 2\pi/aN$, the familiar definition of the wave vector. Equation 4, called a dispersion relation, is one of the most important equations in the theory of vibrations and waves (see plot in Fig. 3). It relates ω and k in an intimate way for wave propagation in periodic structures. It limits the highest frequency of any wave that can propagate to be $\omega = 2\omega_0$ since $\sin(ka/2)$ can never exceed unity. This maximum value $2\omega_0$ is called the "cutoff frequency". In the limiting case where $(\pi/N) = ka/2 \ll 1$, we have $\sin(ka/2) \approx ka/2$. Then we find $\omega = \omega_0 ka$ so a plot of $\omega \;\; vs \;\; k$ is a straight line of slope $\omega_0 a$. Equation 4 governs propagation of sound waves in crystals and quantum mechanical waves of electrons in all materials. It is worthy of your careful inspection and study.

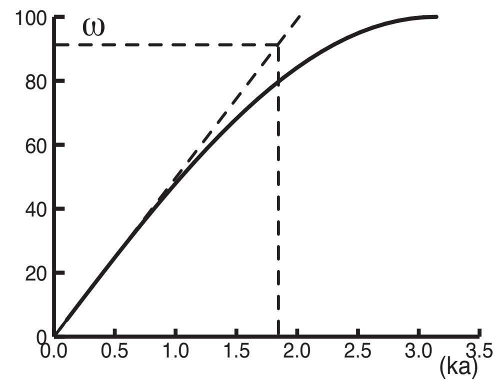

Fig. 3: This is a plot of Eq. 4 for $\omega_0 = 50$ rad/s. Note that for higher values of ω the ka-value is larger than it would be for a straight line (dashed lines), indicating a λ-value that is too small.

3 Procedure

The primary purpose of this experiment is to demonstrate the main properties of wave propagation in a periodic structure. We will see how the dispersion equation (4) above works in the regions $N \gg 1$ and the more interesting region where $N \sim 2$ so that approximating the sine function by its argument is clearly a poor choice.

As stated above, the electrical analog enables easier measurements, and the systems that have been constructed for this experiment consist of 12 LC units. Since approximating 12 by infinity is indeed a very poor approximation, we need to deal with the question of what happens at the end of the line. There are several ways to proceed.

Suppose we consider that the transmission line in Fig. 2 is not infinite, but is indeed very long. A disturbance started at one end propagates along as if it were infinite. But after the disturbance reaches the end, the charge flowing through the last inductor cannot divide between a capacitor and the next inductor, because there **IS** no next inductor. Instead it accumulates on the last capacitor, and eventually is forced back into the last inductor. Thus the disturbance is reflected back toward the beginning of the transmission line.

4 Termination in Characteristic Impedance

We will start by providing a place for that charge to flow so that it doesn't go back through that last inductor. We'll place a resistor across the end of the line to ground, and choose it large enough so we don't short out the last capacitor, but small enough so the charge can drain off that capacitor quickly. The correct choice to give the right time constant for this circuit is $Z_0 = \sqrt{L/C}$ which has the dimensions of Ohms and is called the characteristic impedance. A signal incident on a finite transmission line terminated in its characteristic impedance Z_0 will not be reflected, but will be completely absorbed and dissipated as heat in the resistor. Such a line is indistinguishable from one of infinite length, and thus justifies the use of Eq. 4.

Question 2: Prove that $\sqrt{L/C}$ has the dimensions of ohms.

We begin by injecting a sine wave of very low frequency $f = \omega/2\pi$ so that $\omega/\omega_0 \ll 1$ in Eq. 4 and we can approximate the sine function by its argument. If this were a crystal of lattice spacing a we'd find the phase velocity of the wave $\omega/k = a\omega_0$ and this is the same as the group velocity $d\omega/dk = a\omega_0$. We can measure this speed by measuring the phase change at each of the 12 units, and plotting this phase $vs.$ unit number. If you can determine the number of units required to undergo a phase shift of a full 2π, then this constitutes a "wavelength" in a units. Since you now know both ω and $k = 2\pi/\lambda$, you can calculate the phase velocity.

Question 3: Show that the phase velocity is indeed $\omega/k = a\omega_0$ when we can approximate the sine function by its argument in Eq. 4.

Now repeat this measurement for several increasing values of ω and plot your results. As ω begins to approach the region of ω_0, the sine term in Eq. 4 is approaching 1/2, the low-frequency approximation is no longer valid, and the phase and group velocities are no longer the same. This will be quite apparent from the simple measure of the phase shift even at the first few LC units.

You will find these phase shifts to be larger than at low frequencies simply because the subsequent LC oscillators can't follow the rapid voltage and current changes and so they lag behind a bit more. Then the "wavelength" will correspond to a smaller number of a units. Thus λ will be smaller, k will be larger, and the phase velocity given by $\omega/k = a\omega_0 \sin(\phi/2)/(\phi/2)$ will be smaller. Note that this phase velocity has dimensions "(LC units)/second" and since "LC units" is just a number without dimension, the velocity has dimensions 1/time. If you plot your measured phase velocity vs ω it should look like Fig. 4 which is simply taken from Eq. 4.

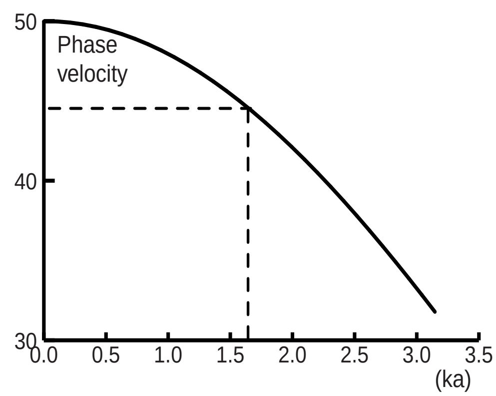

Fig. 4: This is plot of phase velocity given by ω/k, and is taken directly from the dispersion equation, Eq. 4. It is very nearly constant at small k-values (note the vertical scale). It changes by only 10% even for $ka/2 = \pi/4$ (dashed lines).

One of the parameters you can extract from your measurement is $\omega_0 = 1/\sqrt{LC}$. Compare this with the stated values of L and C and comment on the differences. Note that $2\omega_0$ is the cutoff frequency. Calculate it in Hz.

5 Open Circuit and Short Circuit at the End

Instead of a resistor at the end, if we just leave the circuit as it appears in Fig. 2 we have what is called an "open circuit termination." As discussed above, the wave will be reflected, and it will interfere with the incident wave that left the source at a later time. We would have the superposition of counterpropagating waves of the same frequency that we can write as $Y = Y_0[\cos{(\omega t - kx)} + \cos{(\omega t + kx)}]$ which becomes $Y = Y_0 \cos{(\omega t)} \cos{(kx)}$. This interference is constructive if the length of the transmission line corresponds to a quarter wavelength, or in other words, the wavelength is 48 LC units.

Question 4: Show that $Y = Y_0[\cos{(\omega t - kx)} + \cos{(\omega t + kx)}] = Y_0 \cos{(\omega t)} \cos{(kx)}$.

You can easily see this resonance by varying the frequency slowly from a very low value and measuring the voltage at the end unit. This will be a maximum when the desired condition is achieved. If you continue to raise the frequency, the end voltage will decrease to some minimum value, and then rise again. There will be another maximum when the total length of the line is (3/4)λ or λ = 16 units. This behavior will repeat again a 5/4, etc., but as λ begins to approach a small number of units, say one or two, its value will no longer be linear with applied frequency.

This open circuit "standing wave" method is simply another way of finding $k = 2\pi/\lambda$ for various values of ω, and you should make several measurements to determine the dispersion curve again. Be sure to include error bars, and compare your results with those of the first method using the characteristic impedance to terminate the line.

Still another way to find the dispersion curve is to set the resistor at the end of the transmission line to zero ohms, and this is called a "short circuit termination." Also in this case the wave will be reflected because the voltage at the end of the last inductor is pinned to be zero. As above, the reflected wave will interfere with the incident wave that left the source at a later time. Again we would have a standing wave, except that this time the interference is constructive if the length of the transmission line corresponds to a half wavelength, a full wavelength, 3/2 wavelength, etc. You can easily see this resonance and map out the standing wave of period $\lambda/2$.

Question 5: In the two cases of open circuit termination and short circuit termination there are standing waves produced by the reflection. Yet the number of wavelengths (or fractions of wavelengths) along the transmission line at resonance is different. Discuss the two cases, and explain the difference.

6 Hints and Kinks Department

There are several places where you can get tripped up in this experiment. The most significant of these is failure to know how to use the dual trace oscilloscopes, since these are the instruments that will be used for all your measurements. You should be sure that the gain settings and sweep rates correspond to the input signals. Start by connecting the oscillator to channel 1 of the scope only, set the trigger select to channel 1, and then adjust the trigger level until you see a trace. Adjust the scope sensitivity until the sine wave is about half the height of the screen, and adjust the sweep rate so that you see two or three waves across it. Then connect the trigger output from the oscillator to the trigger input of the scope with a second cable, set the trigger select to external, and now adjust the trigger level to get a trace. The trace should be the same as when the scope was triggered on channel 1. Now add a second connection from the output of the oscillator, this one to the counter.

It should tell you the frequency of the oscillator, but it won't necessarily correspond to its dial setting. Now add a third connection from the output of the oscillator, this one to the input end of your transmission line. You may have to readjust the gain on channel 1 to see the same amplitude of the wave.

With this you are ready to take measurements. Connect a probe to channel 2 of the scope and touch its other end to the output of the oscillator. Adjust the gain of channel 2 so that the wave is about the same amplitude as that in channel 1. Now move the probe to the further end of the transmission and see if the waveforms are phase shifted relative to one another. If not, try raising the frequency up to the domain of ω_0. **WARNING:** the counter reads Hz but $1/\sqrt{LC}$ is in radians/s so you have to multiply the counter reading by 2π to compare it with $1/\sqrt{LC}$.

Needless to say, you have to calibrate the time base of the scope carefully so that your measurements have some meaning. You can do this by setting the frequency of the oscillator so that there is an integer number of waves across the scope display, recording the frequency from the counter, and noting how that corresponds to the time base calibration of the scope.

Comment: If we added a resistor R to each loop of the LC circuits, corresponding to the non-zero resistance of the inductor or the leakage of the capacitor, it would act like a friction force and the decay rate corresponding to the γ of a mechanical oscillator would be L/R or RC. In this experiment we are interested only in the case of the ideal circuit elements.